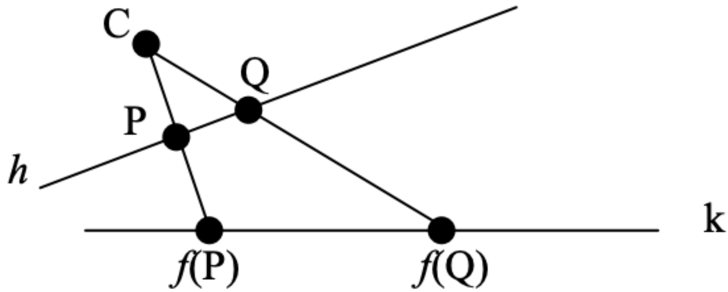
5. PERSPECTIVITIES AND PROJECTIVITIES

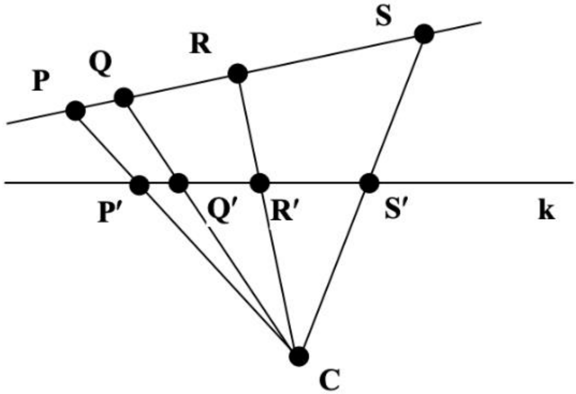
§5.1. Perspectivities

If *h* and *k* are distinct projective lines and C is a projective point on neither, then the **perspectivity** from *h* to *k*, with **centre** C, is the map $f(P) = CP \cap k$.



Theorem 1: Cross ratios are preserved by perspectivities.

Proof: Let f: $h \rightarrow k$ be a perspectivity with centre C. Let P, Q, R, S be four points on *h* with P, Q, R distinct and $S \neq P$. Let $P' = f(P)$, $Q' = f(Q)$, $R' = f(R)$ and $S' = f(S)$.



If $P = \langle \mathbf{p} \rangle$ then, by the Collinearity Lemma, there exists a vector **q** and a scalar λ such that:

$$Q = \langle \mathbf{q} \rangle,$$
$$R = \langle \mathbf{p} + \mathbf{q} \rangle,$$
$$S = \langle \lambda\mathbf{p} + \mathbf{q} \rangle$$

where $\lambda = \Re(P, Q; R, S)$.

Also there exists **c** such that $C = \langle \mathbf{c} \rangle$; $P' = \langle \mathbf{p} + \mathbf{c} \rangle$.

Now C, Q, Q′ are collinear so $Q' = \langle \alpha\mathbf{q} + \beta\mathbf{c} \rangle$ for scalars α, β. Since $Q' \neq C$, $\alpha \neq 0$ so without loss of generality we may take $\alpha = 1$ and $Q' = \langle \mathbf{q} + \beta\mathbf{c} \rangle$.

Since $R' = CR \cap P'Q'$ and

$(\mathbf{p} + \mathbf{c}) + (\mathbf{q} + \beta\mathbf{c}) = (\mathbf{p} + \mathbf{q}) + (1 + \beta)\mathbf{c}$, we have

$$R' = \langle (\mathbf{p} + \mathbf{c}) + (\mathbf{q} + \beta\mathbf{c}) \rangle .$$

Since $S' = P'Q' \cap CS$ and $\lambda (\mathbf{p} + \mathbf{c}) + (\mathbf{q} + \beta\mathbf{c})$

$= (\lambda\mathbf{p} + \mathbf{q}) + (\lambda + \beta)\mathbf{c}$, we have

$$S' = \langle \lambda (\mathbf{p} + \mathbf{c}) + (\mathbf{q} + \beta\mathbf{c} \rangle.$$

So
$$P' = \langle \mathbf{p} + \mathbf{c} \rangle,$$
$$Q' = \langle \mathbf{q} + \beta\mathbf{c} \rangle,$$
$$R' = \langle (\mathbf{p} + \mathbf{c}) + (\mathbf{q} + \beta\mathbf{c}) \rangle,$$
$$S' = \langle \lambda (\mathbf{p} + \mathbf{c}) + (\mathbf{q} + \beta\mathbf{c}) \rangle.$$

Hence $\Re(P', Q'; R', S') = \lambda = \Re(P, Q; R, S)$.

Remarks:

(1) If two points, A and B, and their images are known, one can determine the lines *h* and *k* and the centre of perspectivity, viz. C = A.f(*A*) ∩ B.f(B).

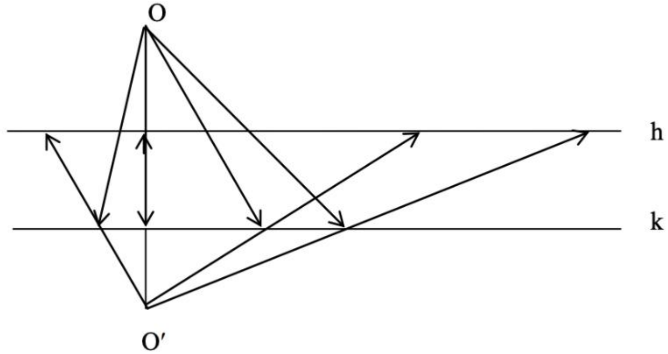
(2) Perspectivities are 1-1 and onto mappings and hence have inverses. Moreover the inverse of a perspectivity is a perspectivity with the same centre.

(3) It might be thought that this follows immediately from the fact that Euclidean cross ratio is identical to Projective cross ratio. However if we embed the Euclidean plane in the Real Projective Plane the corresponding projective lines to P and P′ don't lie in the Euclidean plane.

§5.2. Projectivities

The product of two perspectivities need not be a perspectivity. The simplest way to see this is to take two lines h, k and points, O and O′, that lie on neither of these lines.

If f is the perspectivity from h to k with centre O and g is the perspectivity from k back to h with centre O′ then fg is a map from h to itself. If $O \neq O'$ then fg cannot be the identity map, yet the only perspectivity from a line to itself is the identity map.

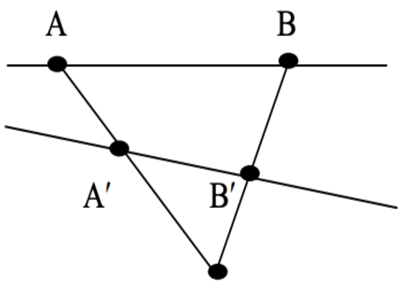


A **projectivity** is the product of a finite number of perspectivities.

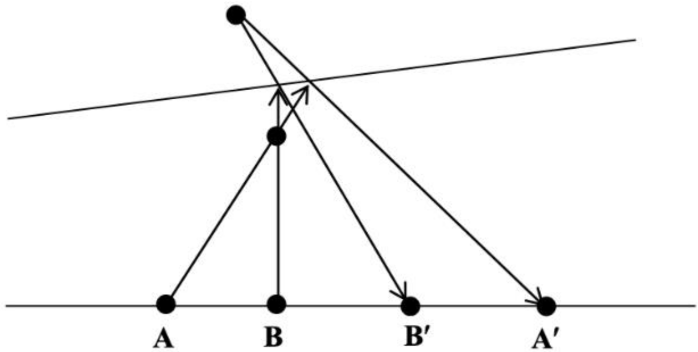
Theorem 2: Projectivities preserve cross ratios.
Proof: This follows immediately from the fact that perspectivities preserve cross ratios.

It follows that given any four points on one line and any four points on another, it is not possible to find a projectivity taking the first four to the second four unless both sets have the same cross ratio.

On the other hand, given any two points A, B on one line and any two points on another, there's a perspectivity (not just a projectivity) which takes A to A′ and B to B′. We simply take AA′ ∩ BB′ as the centre. [If A = A′ or B = B′ there will be more than one possible centre.]



If the lines are the same, a single perspectivity is no longer enough (unless A = A′ and B = B′) but all we need to do is to take A, B off to another line by one perspectivity and back to the original line by another. Thus a 2-step projectivity is sufficient.



The projectivity which does the trick here is not unique because in fact any three points on one line can be sent to any three points on another (or the same) line by a unique projectivity.

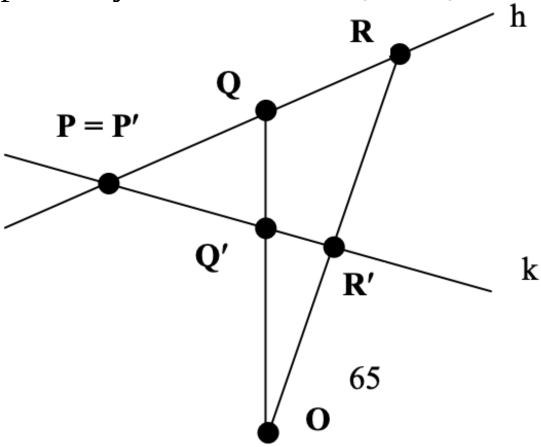
§5.3. The Fundamental Theorem of Projective Geometry

Theorem 3: If h, k are disjoint projective lines and P, Q, R are distinct points on h and P′, Q′, R′ are distinct points on k then there exists a unique projectivity $f : h \rightarrow k$ such that f(P) = P′, f(Q) = Q′ and f(R) = R′. Moreover f is the product of at most 3 perspectivities (or two perspectivities if $h \neq k$).

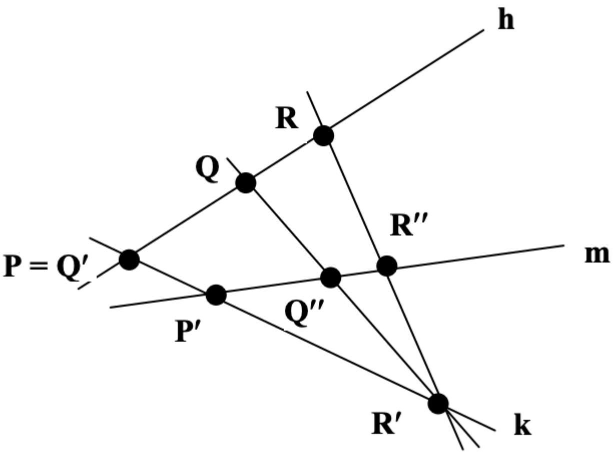
Proof: Case I: $h \neq k$ but one of {P, Q, R} and {P′, Q′, R′} have one point in common.

Without loss of generality we may assume that P = P′ or P = Q′.

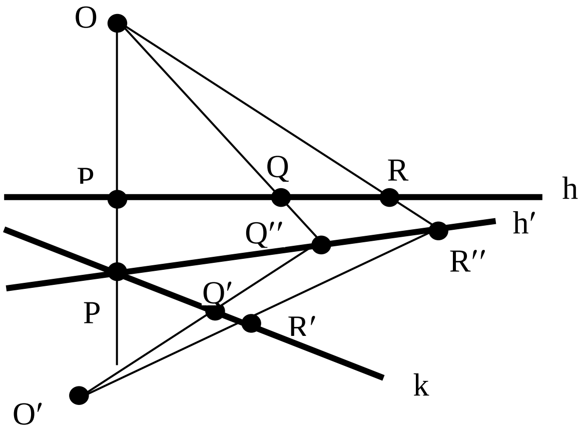
Case IA: P = P′: Any perspectivity from *h* to k must fix P (ie send P to P′) and as we have seen there is a perspectivity which sends Q to Q′ and R to R′.



Case 1B: P = Q ′: Choose any line *m* through P′ different to *k*.
Let Q′′ = QR′ ∩ *m* and R′′ = RR′ ∩ *m*. Then the perspectivity from *h* to *m* with centre R′ takes P to P′, Q to Q′′, R to R′′. By case 1A there's a perspectivity from *m* to *k* that takes Q′′ to Q′ and R′′ to R′. The product of these two perspectivities is the required projectivity.

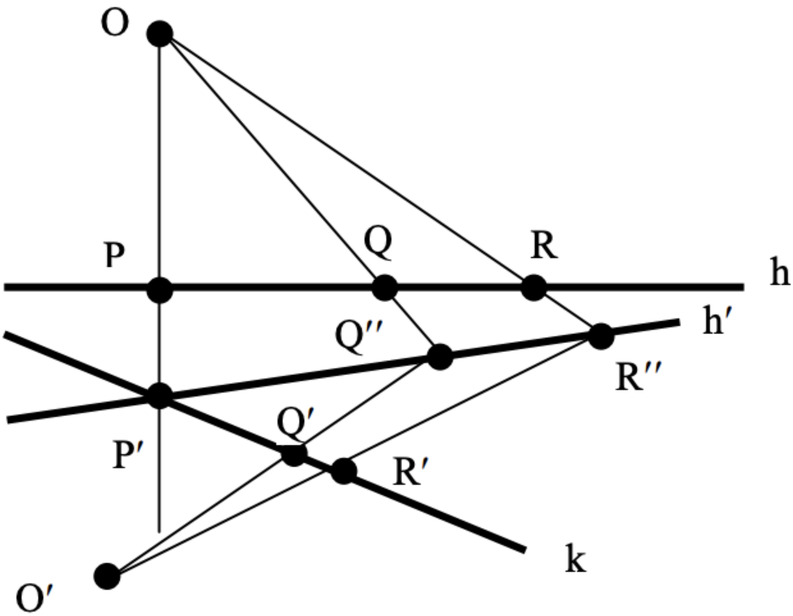


Case 2: *h* ≠ *k* and {P, Q, R} and {P ′, Q ′, R **′} are disjoint.**



We may assume that $h \cap k \neq$ P ′.
(If $h \cap k =$ P′ we proceed with Q′ instead.)

Choose any line h' through P′ with $h' \neq k$ and $h' \neq$ PP′. Choose any point O on PP ′ different from P and P′.



Let f: $h \to h'$ be the perspectivity with centre O.
Then f(P) = P′, f(Q) = Q″, f(R) = R″.
We now need to get from Q″ to Q′ and from R″ to R′. But any perspectivity from h' to k must fix P′ and there is a perspectivity g: $h' \to k$
(with centre O′ = Q′Q″ ∩ R′R″) which sends Q″ to Q′ and R″ to R′. Thus fg: $h \to k$ is a suitable (2-step) projectivity.

Case III: *h* = *k*
In this case we simply take P, Q, R off to another line by any perspectivity and proceed as in Case II.

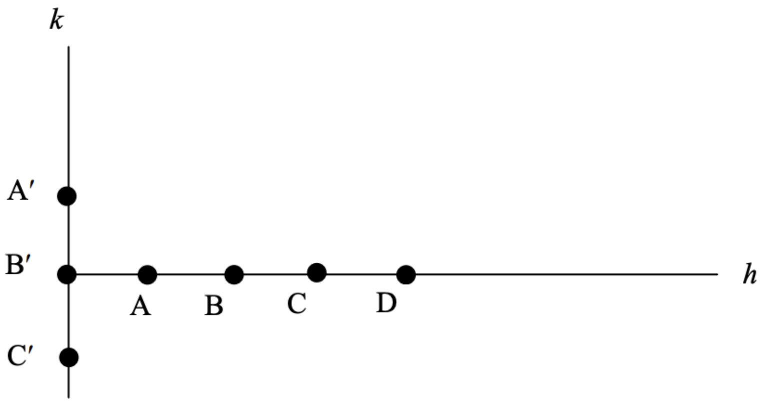
The uniqueness of the projectivity follows from the fact that once the images of three points P, Q, R are given all other images are determined because of the fact that cross-ratios must be preserved. If X is any point on *h* then *f*(X) must be that unique point X′ on *k* such that $\Re(P', Q'; R', X') = \Re(P, Q; R, X)$.

WARNING: The fact that the projectivity is unique doesn't mean that its factorization into perspectivities is unique.

Example 2: Find a projectivity *f* taking A to A′ etc where:

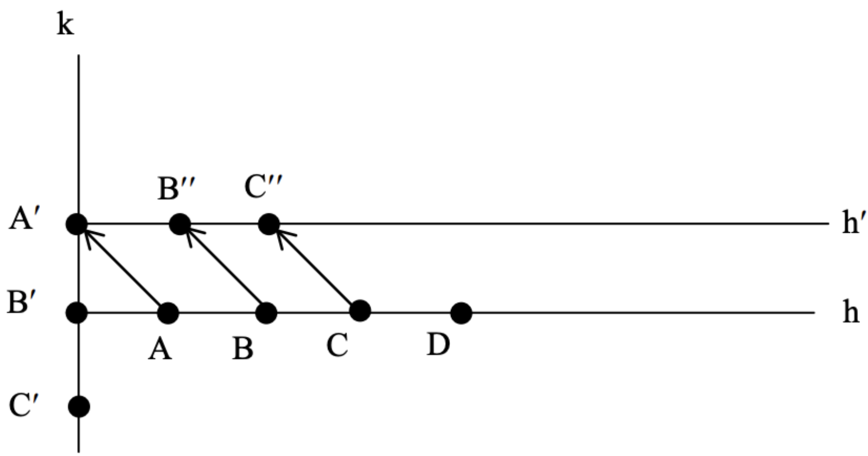
A = (1, 0) A′ = (0, 1)
B = (2, 0) B′ = (0, 0)
C = (3, 0) C′ = (0, –1).

If D = (4, 0), find f(D).



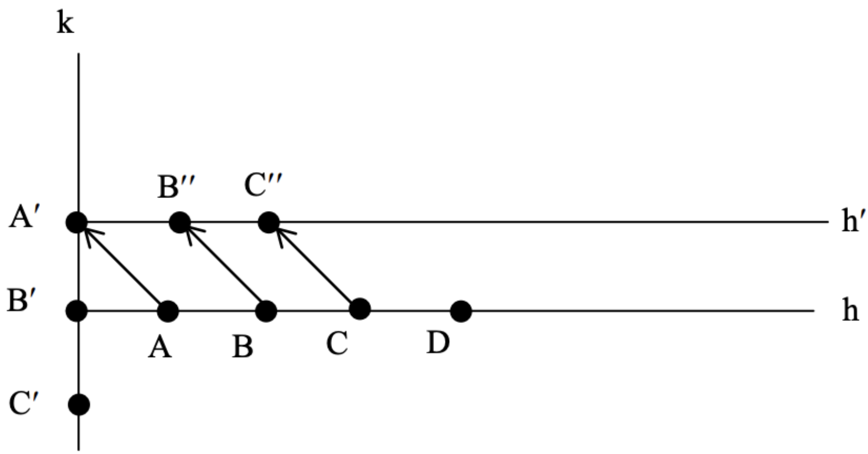
Solution: Here the points are in a Euclidean plane, regarded as being embedded in the real projective plane. We follow Case II of the proof. Choose h' to be the line $y = 1$. Choose O to be the ideal point on the line AA′. (We could have chosen any ordinary point on the line, apart from A or A′ themselves but using an ideal point will give us good practice in viewing the Euclidean plane as embedded in the real projective plane.)

Let $f: h \to h'$ be the perspectivity with centre O.
Then B″ = f(B) = (1, 1) and
C″ = f(C) = (2, 1).



Then take as $g: h' \to k$ the perspectivity with centre B′B″ ∩ C′C″. Since the lines are parallel in the Euclidean Plane their intersection in the projective plane is the ideal point O′ on the line $y = x$.

Then fg: $h \to k$ is the required projectivity.



Now f(D) = (3, 1) and so (fg)(D) = g(f(D)) = (0, –2).

§5.4. Harmonic Conjugates

If A, B, C, H are collinear points then H is called the **harmonic conjugate** of C, relative to A, B, if

$$\Re(A, B; C, H) = -1.$$

Construction for the Harmonic Conjugate

Theorem 4: Let A, B, C be distinct collinear points.

(1) Choose D not on AB

(2) Choose E on BD distinct from B and D themselves.

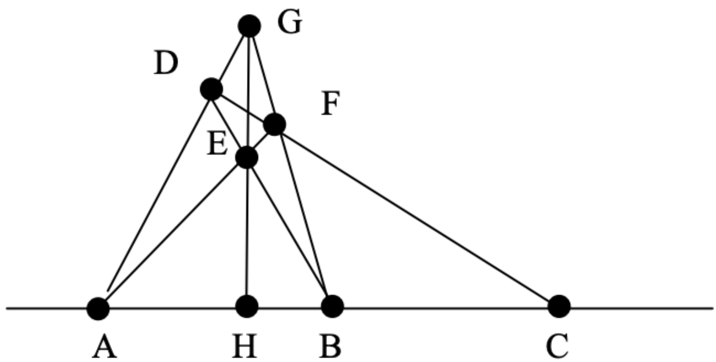
(3) Let F = AE ∩ CD.

(4) Let G = AD ∩ BF.

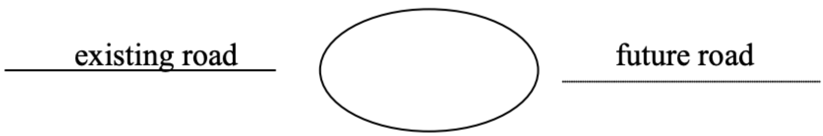
(5) Let H = AB ∩ EG.

Then H is the harmonic conjugate of C relative to A, B.

Proof: This is left as an exercise. [Apply the Collinearity Lemma to the lines ABC and BED and then find vectors to represent all the other points. The harmonic conjugate, H, should be ⟨**a** – **b**⟩ = ⟨–**a** + **b**⟩.]



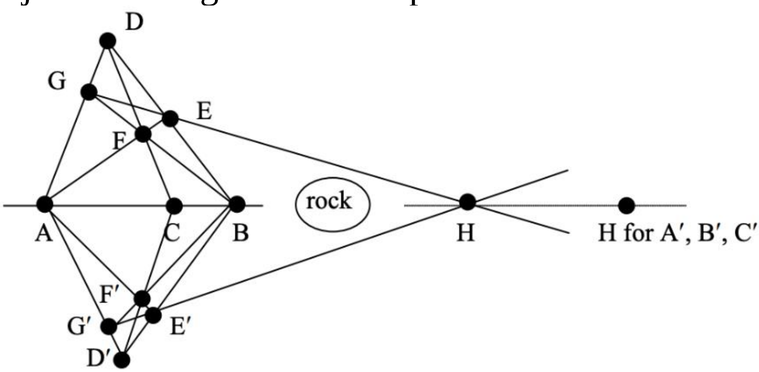
Example 3: There's a straight road which ends at the base of Uluru (Ayers Rock) in Central Australia. It is desired to continue the road on the other side of the rock so that both portions are in the same straight line. How can this be done using only standard surveying equipment which can effectively draw a straight line between two points (provided the rock is not in the way)?



Solution: Take points A, B, C on the existing road and carry out the Harmonic Conjugate construction to the point where the line EG is drawn. Repeat the construction with different choices of D, E named D′ and E′ to obtain the line E′G′. Then EG and E′G′ will intersect at the Harmonic Conjugate of A, B relative to C and this point will lie on the line AB of the road.

If the choices are suitably made this point will lie on the far side of the rock and none of the construction lines will pass through the rock and so will be possible. Repeat all of this again, using different points A, B, C on the existing road and a second point on the line of the road can be obtained (again on the far side if suitable choices are made). You'll now have two points on the line AB on the opposite side of the rock to the existing road and these can now be joined.

In order to get H to lie on the far side of the rock we'd need to take C between A and B. Choosing it to be the midpoint would result in H being the ideal point on AB, which would not suit a surveyor. On the other hand choosing C too close to B might result in H being too close to B and hence lie underneath the rock. So a choice of C just to the right of the midpoint seems to be suitable,



EXERCISES FOR CHAPTER 5

Exercise 1: Suppose *m* is the line $y = 2x + 1$ in the real affine plane, and *n* is the line
$y = 6 – x$. Let A be the point (1, 3) on *m* and let P be the point (*2*, *1*). Let $f:m\to n$ be the perspectivity from *m* to *n* with centre P. Find *f*(A).

Exercise 2: Suppose *m* is the line $y = 2x + 1$ in the affine plane over $\mathbb{Z}_5$, and let *n* be the line $y = 2$. Let P be the point (1, 1). Let $f:m\to n$ be the perspectivity from *m* to *n* with centre P.
(i) If A is the point (2, 0), find *f*(A);
(ii) If B is the point (4, 1), find *f*(B).

Exercise 3: Suppose O = $\langle(1, 1, 1)\rangle$, A = $\langle(2, 1, 0)\rangle$, B = $\langle(3, 4, 3)\rangle$, P = $\langle(2, 1, 3)\rangle$ and
X = $\langle(3, 2, 1)\rangle$ in the real projective plane. Let *f* be the projectivity from OA to OB with centre P. Find *f*(X).

Exercise 4: Let *m* be the line $y = x + 1$, let *n* be the line $x = 5$ and let *r* be the line $y = 2 – 3x$ in the real affine plane. Let O = (1, 0), Q = (0, 1) and X = (3, –2). Let *f* be the perspectivity from *m* to *n* with centre O and *g* is the perspectivity from *n* to *r* with centre Q and $h = fg$. Find *h*(X).

Exercise 5: In the real projective plane, find the centre of the perspectivity *f* that takes P(0, 1) to P′(0, 2) and Q(0, 2) to Q′(1, 0). Does *f* take R(0, 3) to R′(3, 0)?

Exercise 6: In the real projective plane, find a projectivity f that takes P(0, 1) to P′(0, 2), Q(0, 2) to Q′(1, 0) and R(0, 3) to R′(3, 0). Is there such a projectivity that also takes S(0, 4) to S′(5/2, 0)? Illustrate your solution with a diagram.

Exercise 7:

(i) If $\Re(A, B; C, D) = \lambda$, find $\Re(B, C; D, A)$ in terms of λ.

(ii) If A, B, C, D are distinct collinear points, for what value or values of λ (if any) does there exist a projectivity f from the real line to itself such that:

$f(A) = B$, $f(B) = C$, $f(C) = D$, $f(D) = A$?

(iii) Let A, B, C, D be the points 0, 1, 4, and –2 respectively on the real line. Use the Euclidean interpretation of cross ratio to calculate $\Re(A, B; C, D)$.

(iv) Construct a projectivity, from the real line to itself, such that:

$$A \to B,\ B \to C,\ C \to D,\ D \to A.$$

(Describe the construction in words, as a sequence of perspectivities, and illustrate your construction with a diagram.)

Exercise 8:

(a) Let A = (2, 1), B = (–2, 3), C = (0, 2) and D = (6, –1) be points in the Euclidean plane. Find $\Re(A, B; C, D)$.

(b) Let h denote the line $y = 2 - (½)x$ on which the points in (a) lie. Exhibit a projectivity from h to h under which $A \to C$, $B \to A$, $C \to B$.

(c) Determine the image of D under the projectivity in part (b).

Exercise 9: In the Euclidean plane extended by ideal points and the ideal line let A = (0, 0), B = (1, 0) and let C be the ideal point on the x-axis. Carry out a construction to find the harmonic conjugate of C relative to A and B.

SOLUTIONS FOR CHAPTER 5

Exercise 1: The line joining P to A is $y = -2x + 5$.
It cuts $y = 6 - x$ when $x = -1$.
So $f(\text{A}) = (-1, 7)$.

Exercise 2:
(i) The line joining P to A is $y = 4x + 2$. It cuts $y = 2$ when $x = 0$. So $f(\text{A}) = (0, 2)$.
(ii) The line joining P to B is $y = 1$. It cuts $y = 2$ at the ideal point on that line.
Hence $f(\text{A})$ is the ideal point on the lines $y = c$, for constants c.

Exercise 3: $\text{PX} = \langle(2, 1, 3) \times (3, 2, 1)\rangle^{\perp} = \langle(-5, 7, 1)\rangle$.
$\text{OB} = \langle(1, 1, 1) \times (3, 4, 3)\rangle^{\perp} = \langle(-6, -6, 12)\rangle$.
Hence $f(\text{X}) = \langle(-5, 7, 1) \times (-6, -6, 12)\rangle^{\perp}$
$= \langle(90, 54, 72)\rangle = \langle(5, 3, 4)\rangle$.

Exercise 4: The line OX is $y = -x + 1$.
This cuts $x = 5$ at $(5, -4)$, so $f(X) = (5, -4)$.
The line Q f(X) is $y = -x + 1$. This cuts $y = 2 - 3x$ at $(\frac{1}{2}, \frac{1}{2})$, so $h(X) = g(f(X)) = (\frac{1}{2}, \frac{1}{2})$.

Exercise 5: The centre is $PP' \cap QQ' = (1/3, 5/6)$. The line joining this point to R has equation:

$$y = (-13/2)x + 3.$$

It cuts $P'Q'$ (the x-axis) at $(6/13, 0)$. This is f(R) but not R'.

Exercise 6: Let h be the y-axis and let k be the x-axis. Take h' (any line through P' not equal to k) to be the line $x = 2$. The line PP' is $y = 1 - (\frac{1}{2})x$ so choose O (any point on PP' except those two points themselves) to be $(1, \frac{1}{2})$.

The line joining O to $(0, b)$ on h is $y = (\frac{1}{2} - b)x + b$ and it cuts h' when $x = 2$ and $y = 1 - b$. So the images of P, Q, R and S under the perspectivity from h to h' with centre O are: P', $Q''(2, -1)$, $R''(2, -2)$, $S''(2, -3)$.

We now project from h' to k.
The centre is $O' = Q'Q'' \cap R'R''$.
This is the intersection of the lines $y = -x + 1$ and $y = 2x - 6$. So O' is $(7/3, -4/3)$.

Projecting R'' through O' onto k we get $(13/5, 0)$. This is the image of R under the projectivity, and so is not R'.

Could there be another projectivity that works (after all we made some choices in constructing the one we did)?

$\Re(P, Q; R, S) = \dfrac{PR/QR}{PS/QS} = \dfrac{4}{3}$, while

$\Re(P', Q'; R', S') = \dfrac{P'R'/Q'R'}{P'S'/Q'S'} = \dfrac{3}{2}$.

Since projectivities preserve cross ratios there can be no projectivity that takes each of P, Q, R, S to P′, Q′, R′ and S′ respectively.

Exercise 7:

(i) $\Re(B, A; C, D) = 1/\lambda$.

$\therefore\ \Re(B, C; A, D) = 1 - 1/\lambda = (\lambda - 1)/\lambda$.

$\therefore\ \Re(B, C; D, A) = \lambda/(\lambda - 1)$..

(ii) We must have $\dfrac{\lambda}{\lambda - 1} = \lambda$ so $\lambda = 2$ (NB $\lambda \neq 0$ since the points are distinct.)

(iii) $\Re(0, 1; 4, -2) = \dfrac{4/3}{-2/-3} = 2$.

(iv) By (i) and (ii) $\Re(A, B; C, D) = \Re(B, C; D, A)$ and so there exists such a projectivity.

(1) Take the perspectivity from the *x*-axis to the line $y = 1$ with centre the ideal point on vertical lines.

This takes A, B, C, D to A′ = (0, 1), B′ = (1, 1), C′ = (4, 1), D′ = (–2, 1) respectively.

(2) Take the perspectivity from the line $y = 1$ to the line $x = 1$ with centre the ideal point on lines with slope –1. This takes A′, B′, C′, D′ to A″ = (1, 0), B″ = (1, 1), C″ = (1, 4), D″ = (1, –2) respectively.

(3) The line B″C is y = (–1/3)x + 4/3.
The line C″D is y = (4/3)x + 8/3.
They intersect at (–4/5, 8/5). So we take the perspectivity from the line x = 1 to the x-axis again, with centre (–4/5, 8/5).

This is the required projectivity. The line AD″ is y = – (1/2)x. Since this passes through (–4/5, 8/5) this verifies that *f*(D) is indeed A.

Exercise 8:(a) To avoid awkward square roots with distances we can project the points onto the *x*-axis. This amounts to taking a perspectivity whose centre is the ideal point on the *y*-axis. In other words, we can simply work with the *x*-coordinates.

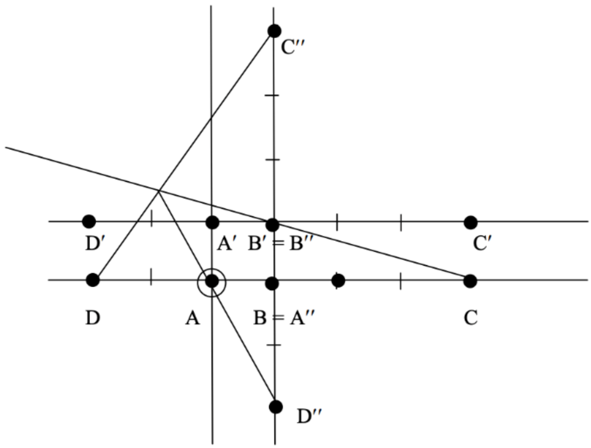
$$\Re(A, B; C, D) = \frac{AC/BC}{AD/BD} = -2.$$

(b) Using the perspectivity described above we take the four points off onto another line (the x-axis) to A′(2, 0), B′(–2, 0), C′(0, 0) and D′(6, 0).

We must now take A′ → C, B′ → A and C′ → B. We take $y = 2$ as our line through C and we choose O to be (1, 1).

Taking the perspectivity from the *x*-axis to *y* = 2 with centre O, the point (*t*, 0) maps to (2 – *t*, 2).

So A′ maps to C, B′ maps to B″(4, 2), C′ maps to C″(2, 2).

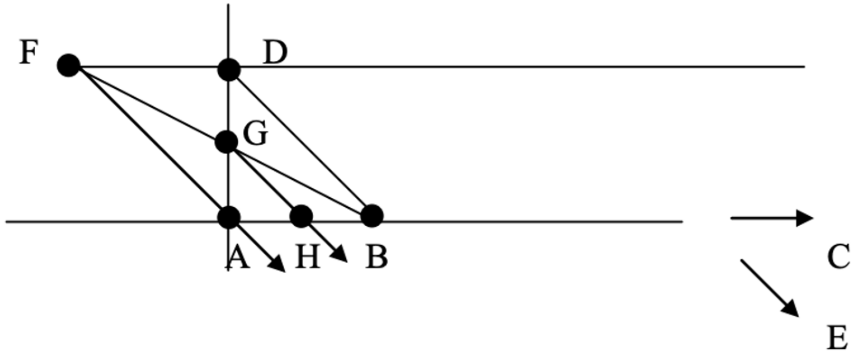


Now O″ = B″A ∩ C″B = (10/3, 5/3).
So we take the perspectivity from the line *y* = 2 to the *x*-axis with centre (10/3, 5/3). The product of these three perspectivities is the required projectivity. To summarise, these perspectivities are:

from line	**to line**	**with centre**
y = 2 – (½)*x*	*x*-axis	ideal point on *y*-axis
x-axis	*y* = 2	(1, 1)
y = 2	*y* = 2 – (½)*x*	(10/3, 5/3)

(c) Under this projectivity:
D → (6, 0) → (–4, 2) → (2/5, 9/5).

Exercise 9:



Choose D = (0, 1) (not on AB). Choose E = ideal point on BD. E

F = AE $\cap$ CD = (–1, 1), G = AD $\cap$ BF = (0, ½),

H = AB $\cap$ EG = (½, 0).